



Geometrically Exact Finite Element Formulation for Tendon-Driven Continuum Robots

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ABSTRACT Tendon-driven continuum robots achieve continuous deformations through the contraction of tendons embedded inside the robotic arms. For some continuum robots, the constant curvature assumption-based kinematic modeling can be accurate and effective. While for other cases, such as soft robots or robot-environment interactions, the constant curvature assumption can be inaccurate. To model the complex deformation of continuum robots, the geometrically exact beam theory (may also be called the Cosserat rod theory) has been used to develop computational mechanics models. Different from previous computational models that used finite difference schemes for the spatial discretization, here we develop a three-dimensional geometrically exact beam theory-based finite element model for tendon-driven continuum robots. Several numerical examples are presented to show the accuracy, efficiency, and applicability of our new computational model for tendon-driven continuum robots.

KEY WORDS Continuum robots, Cosserat rod model, Geometrically exact beam, Finite element method

1. Introduction

Continuum robots are a type of robotic arms that can bend continuously along their length direction to produce robotic motions similar to those of elephant trunks, octopus tentacles, or snakes [1, 2]. The compliance and dexterity of continuum robots make them better suited for special tasks such as navigating in geometrically confined environments. A number of continuum robots have been developed in the past, for example, a tendon-driven metallic continuum manipulator designed to function as a steerable catheter [3], a concentric-tube continuum manipulator [4], a tendon-driven soft robot arm mimicking elephant trunk [5], and a pneumatic-driven soft robotic arm [6]. In addition, continuum robots made of soft materials can further enhance their flexibility and compliance, which have attracted considerable research interest recently [7–13]. Note that the present work is focused on the computational mechanics modeling of a particular type of continuum robot: the tendon-driven continuum robots made of hard materials. As shown in Fig. 1a, the tendon is embedded inside the rod-shaped

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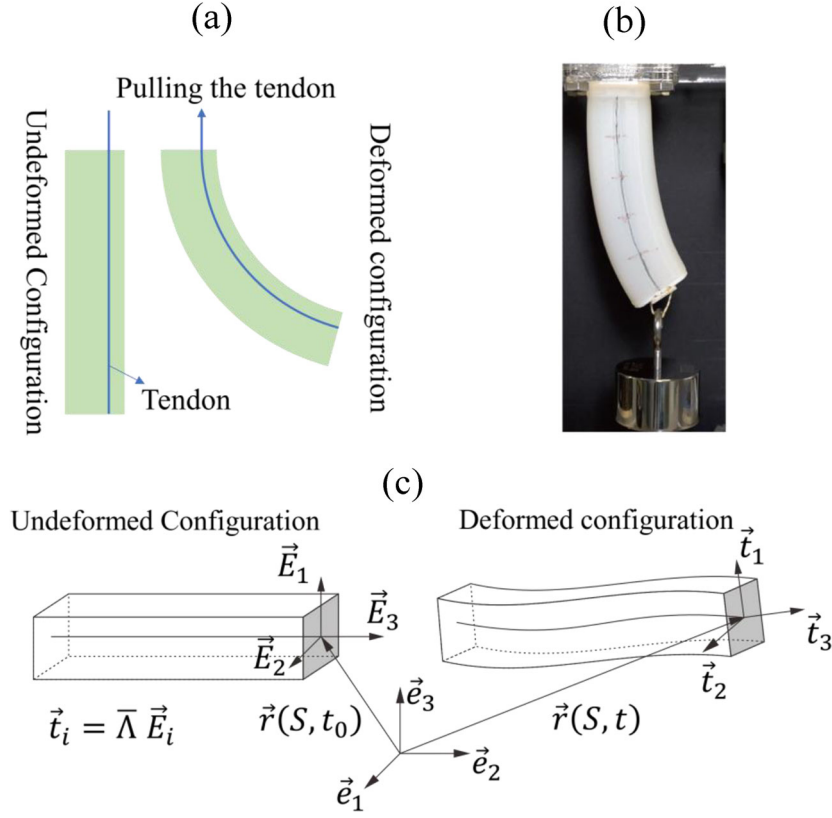


Fig. 1. **a** A schematic showing that tendon-pulling/contraction causes the robotic arm to bend; **b** bending deformation of a soft tendon-driven continuum robot; **c** rigid body rotation of infinitesimal rigid disks from the undeformed configuration to the deformed configuration

robotic arm with a distance off to the line of the centroids, and the contraction/pulling of the tendon causes the arm to bend.

In theory, continuum robots have an infinite number of degrees of freedom due to the continuous deformation along the length direction. A commonly used analytical model for continuum robots is the constant-curvature kinematics model that assumes a robot arm bends into an arc with a constant curvature [2]. This assumption significantly simplifies the kinematic modeling by reducing the infinite number of degrees of freedom to a few. However, the constant-curvature assumption becomes inaccurate in certain situations [6]: (1) the external forces exerted on the robotic arm (e.g., contact force and gravity) are sufficiently large to cause continuously varying curvature, (2) the tendon paths within the robotic arm are not parallel to the centroidal line of the arm, (3) the mechanical and geometric properties of the arm are not uniform along the length direction, etc.

To improve the accuracy of the mechanical modeling of continuum robotic manipulators, researchers have applied the beam theory, especially the geometrically exact beam (GEB) theory [14] to model continuum robotic arms [4, 6, 15, 16]. In the field of continuum robots, the GEB theory is often referred to as the Cosserat rod theory. Here in a narrow sense, “geometrically exact” means that, for arbitrary deformation of beams, only the kinematic assumption of rigid cross sections is made. The GEB model is an extension of the classical beam models of Kirchhoff, Love, and Reissner to the 3D large deformation scenario that can model a combination of different deformation modes including bending, shearing, torsion, and axial extension [17–20]. Therefore, the GEB model is more accurate than the constant-curvature model for predicting the deformed configuration of flexible or soft continuum robots [6]. Previously, the spatial discretization methods utilized in GEB theory-based computational models by researchers in robotics [15] were mainly based on finite difference schemes. In some other studies

on the GEB theory-based modeling of continuum robots, only two-dimensional planar problems were considered [6]. To the best of the authors' knowledge, none of the previous studies have applied the GEB theory-based finite element method (FEM) to continuum robotic manipulators, despite that the FEM is prevalent in structural mechanics.

In his seminal work [18], Simo derived the geometrically exact kinematic relations for the finite strain measures using the 3D continuum mechanics methodology. Based on the GEB theory, Simo and Vu-Quoc developed the finite element model of beams [21]; and furthermore, they incorporated the torsion-warping deformation of the cross section in the beam model [22]. After that, the GEB theory-based finite element modeling has attracted considerable attention because of its high accuracy and efficiency [17, 23–28]. Crisfield and Jelenic [23] recognized that the interpolation scheme for the rotation parameters in Simo and Vu-Quoc's numerical implementation [21] violates the rotation-related objectivity. They also proposed and implemented a new interpolation scheme that preserves the objectivity and path-independency [24]. Thereafter, many other computational methods that preserve the objectivity and path-independency have been developed [24, 29–32]. However, the interpolation scheme of Simo and Vu-Quoc [21] can produce accurate numerical results as demonstrated in their paper as well as in Jelenic and Crisfield's paper [24], where the authors demonstrated that, when the element size is relatively small, the error induced by the non-objective interpolation scheme becomes negligible.

In the present study, we apply the GEB theory-based FEM to the computational mechanics modeling of tendon-driven continuum robots. Due to the relatively simple finite element formulation and implementation of Simo and Vu-Quoc [21], we adopt their method as the basis for formulation of the GEB theory-based finite element model of tendon-driven continuum robotic manipulators. In the following sections, the finite element model of the tendon-driven continuum robots will be presented and the performance of the model will be examined by several numerical examples.

2. Mechanics Model of Tendon-Driven Continuum Robots

2.1. Configuration Description of the Robotic Arm

Here, we use the following conventions for notation: (1) A variable with an arrow above it represents a vector, e.g., $\vec{x}$. (2) A variable with a bar above it represents a second-order or higher-order tensor, e.g., $\bar{\Lambda}$. (3) A bold symbol, e.g., $\mathbf{\Lambda}$, represents the matrix notation of a vector or tensor. Unless specifically mentioned, the elements of a matrix are the components of corresponding vector or tensor in the fixed spatial Cartesian base vectors $\{\vec{e}_i\}_{i=1,2,3}$.

As shown in Fig. 1b, a tendon-driven continuum robotic arm changes its shape from an undeformed configuration to a current curved configuration, upon the tendon contraction inside the arm. In the present work, unless specifically mentioned, the robotic arm is assumed to be straight and uniform along the length direction when it is undeformed, and the tendon path in the undeformed configuration is also straight and parallel to the line of centroids of the arm. For convenience, we sometimes use the reference configuration or material configuration as the undeformed configuration.

With the kinematic assumption of the rigid cross section, mathematically, the deformation of the original flexible robotic beam is simplified as the motions of a string of rigid disks with infinitesimal thickness, as illustrated in Fig. 1c. Therefore, for the rigid disks constructing a robotic arm, the motion as a function of time t can be fully described by the position vector $\vec{r}(S, t)$ of the line of centroids of the beam and the moving frame $\{\vec{t}_I(S, t)\}_{I=1,2,3}$ attached to the rigid disks. Here, S is the arc length of the centroidal line in the reference (unstressed) configuration and is used as the coordinate for infinitesimal disk, the basis vectors $\vec{t}_1$ and $\vec{t}_2$ are within the plane of cross sections and $\vec{t}_3$ is normal to the cross section ($\vec{t}_1$, $\vec{t}_2$, and $\vec{t}_3$ are fixed with the rigid cross sections). It must be noted that for soft continuum robots undergoing large deformations, the areas and shapes of the cross sections may vary substantially, thereby, the rigid cross-sectional assumption may be violated in those cases. So the present model is better suited for the flexible continuum robots made of hard materials that undergo large rotation but small strain.

The basis vectors of the reference configuration of the beam, denoted by $\left\{\vec{E}_I(S)\right\}_{I=1,2,3}$, can be considered as the values of $\left\{\vec{t}_I(S,t)\right\}_{I=1,2,3}$ at the initial instant t_0 , that is,

$$\vec{E}_I(S) = \vec{t}_I(S, t_0) \quad (1)$$

Therefore, the rigid-body rotation of the infinitesimally thin rigid disk from material frame $\left\{\vec{E}_I(S)\right\}_{I=1,2,3}$ in the undeformed configuration to the moving frame $\left\{\vec{t}_I(S,t)\right\}_{I=1,2,3}$ in the current configuration can be represented by an orthogonal transformation tensor $\bar{\Lambda}(S,t)$ defined as

$$\bar{\Lambda} \equiv \vec{t}_I \otimes \vec{E}_I \quad (2)$$

where $\otimes$ denotes the tensor product, and Einstein's summation convention is applied here for the dummy index I . Thus, we have $\vec{t}_I = \bar{\Lambda} \vec{E}_I$.

The position vector of a material particle on the tendon can be written as:

$$\vec{\chi}(S,t) \equiv \vec{r}(S,t) + \sum_{\eta=1}^2 \zeta_{\eta} \vec{t}_{\eta}(S,t) = \vec{r}(S,t) + \vec{\zeta}(S,t) \quad (3)$$

where (ζ_1, ζ_2) are the components of the relative position vector of the tendon with respect to $\vec{t}_1$ and $\vec{t}_2$ in the cross-sectional plane at S . For the straight arm and straight tendon in the undeformed configuration, (ζ_1, ζ_2) are constant with arc length S and time t . The configuration of the arm, including the tendon position, is fully determined by $\left\{\vec{r}(S,t), \bar{\Lambda}(S,t)\right\}$.

2.2. Strain Measures, Stress Resultants, and Couples

As shown in Fig. 2, the cross section is composed of two regions: the beam region and the tendon. Therefore, there are two sets of stress resultants and couples exerted on the cross section.

One set includes $\vec{f}_b$ and $\vec{m}_b$ that are related to beam deformation; the other set involves $\vec{f}_c$ and $\vec{m}_c$ that are related to tendon tension. The material strain measures conjugating to the material stress resultant $\vec{F}_b \equiv \bar{\Lambda}^T \vec{f}_b$ and material stress couple $\vec{M}_b \equiv \bar{\Lambda}^T \vec{m}_b$ have been derived by Simo [18] as

$$\begin{aligned} \vec{\Gamma} &\equiv \bar{\Lambda}^T \left(\frac{\partial \vec{r}}{\partial S} - \vec{t}_3 \right) \\ \vec{\kappa} &\equiv \left(\bar{\Lambda}^T \frac{\partial \bar{\Lambda}}{\partial S} \right)^{\vee} \end{aligned} \quad (4)$$

where $\vec{\Gamma}$ and $\vec{\kappa}$ have been named the translational and rotational strain measures [24]. Here, notation $()^{\vee}$ denotes the axial vector of a skew-symmetric tensor and symbol $()^{\wedge}$ denotes the corresponding skew-symmetric tensor of the axial vector. The derivative of the moving basis vector with respect to S is

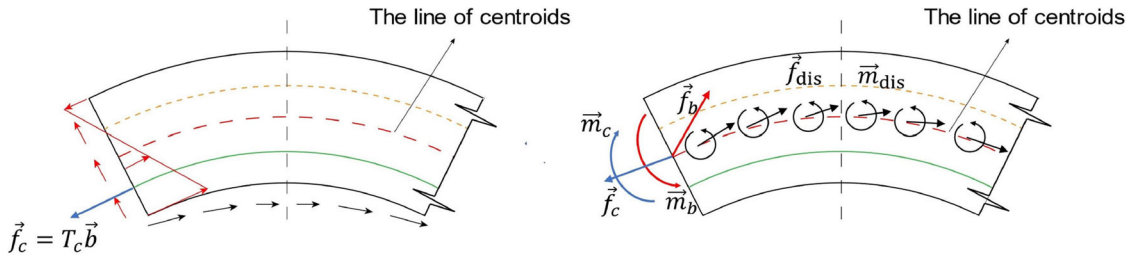


Fig. 2. Stress distribution on the cross section (left) and the equivalent stress resultant and couple (right)

$$\frac{\partial}{\partial S} \vec{t}_I = \bar{\Omega} \vec{t}_I \quad (5)$$

where $\bar{\Omega} = \bar{\Lambda} \vec{\kappa}^\wedge \bar{\Lambda}^T$ is a skew-symmetric tensor field.

For elastic materials, the constitutive relations can be written as

$$\begin{aligned} \vec{F}_b &= \frac{\partial W(S, \vec{\Gamma}, \vec{\kappa})}{\partial \vec{\Gamma}} \\ \vec{M}_b &= \frac{\partial W(S, \vec{\Gamma}, \vec{\kappa})}{\partial \vec{\kappa}} \end{aligned} \quad (6)$$

where W is the stored energy function. The material elasticity tensor can be defined as

$$\mathbf{C} \equiv \begin{bmatrix} \frac{\partial \vec{F}_b}{\partial \vec{\Gamma}} & \frac{\partial \vec{F}_b}{\partial \vec{\kappa}} \\ \frac{\partial \vec{M}_b}{\partial \vec{\Gamma}} & \frac{\partial \vec{M}_b}{\partial \vec{\kappa}} \end{bmatrix} \quad (7)$$

where, for simplicity, $\mathbf{C}$ is often assumed constant and diagonal in the material basis $\{\vec{E}_i\}$:

$$\mathbf{C}|_{\vec{E}_i} = \text{diag} [GA_1 \ GA_2 \ EA \ EI_1 \ EI_2 \ GJ] \quad (8)$$

Here, GA_1 and GA_2 denote the shear stiffness along axes $\vec{t}_1$ and $\vec{t}_2$, respectively; EA is the axial stiffness; EI_1 and EI_2 are the principal bending stiffness relative to $\vec{t}_1$ and $\vec{t}_2$, respectively; GJ is the torsional stiffness of the arm; and symbol $|_{\vec{E}_i}$ denotes the reference frame in which the components of a tensor is defined. Note that the linear constitutive model in Eq. (8) is limited to continuum robots that undergo large rotation but small strain, such as the flexible continuum robots made of hard materials.

The tendon tension-associated stress resultant and stress couple in the spatial form can be defined based on the tangent direction $\vec{b}$ of the tendon and the given magnitude of the tendon tension T_c :

$$\begin{aligned} \vec{f}_c &= T_c \vec{b} \\ \vec{m}_c &= \vec{\zeta} \times \vec{f}_c \end{aligned} \quad (9)$$

where $\vec{b} = \frac{\vec{x}'}{|\vec{x}'|}$, with the prime representing the derivative with respect to the reference arc length S .

In this work, $T_c(S) \equiv T_c$, which means we ignore the friction between the cable and the channel wall. The stress resultant and stress couple in the material form can be written as

$$\begin{aligned} \vec{F}_c &\equiv \bar{\Lambda}^T \vec{f}_c \\ \vec{M}_c &\equiv \bar{\Lambda}^T \vec{m}_c \end{aligned} \quad (10)$$

Different from the previous work [4] where the tendon was considered as an external object that exerts forces on the robotic arm, we treat the tendon as a part of the arm (see Fig. 2). Therefore, the contact forces exerted on the robotic arm by the tendon, which are internal forces, do not appear in the free body diagram, nor in the equilibrium equation of the robotic arm (i.e., Eq. (11)).

2.3. The Equilibrium Equations and the Weak Form

In the present work, we only consider elastostatic problems. The equilibrium equations of the robotic arm can be obtained by adding the tendon-related force and moment to the beam equations [18] as:

$$\begin{aligned} \frac{d\vec{f}_b}{ds} + \frac{d\vec{f}_c}{ds} + \vec{f}_{\text{dis}} &= 0 \\ \frac{d\vec{m}_b}{ds} + \frac{d\vec{m}_c}{ds} + \frac{d\vec{r}}{ds} \times (\vec{f}_b + \vec{f}_c) + \vec{m}_{\text{dis}} &= 0 \end{aligned} \quad (11)$$

where $\vec{f}_{\text{dis}}$ and $\vec{m}_{\text{dis}}$ are the distributed force and moment per unit reference arc length, respectively. For the convenience of performing linearization, the weak form of these equations is written in the material configuration and in matrix form as

$$G \equiv \int_0^L \left[(\mathbf{F}_b + \mathbf{F}_c)^T \mathbf{\Lambda}^T ((\delta \mathbf{u})' - (\delta \boldsymbol{\vartheta})^\wedge \mathbf{r}') + (\mathbf{M}_b + \mathbf{M}_c)^T \mathbf{\Lambda}^T (\delta \boldsymbol{\vartheta})' \right] dS - \int_0^L \left(\mathbf{f}_{\text{dis}}^T \delta \mathbf{u} + \mathbf{m}_{\text{dis}}^T \delta \boldsymbol{\vartheta} \right) dS = 0 \quad (12)$$

where $\delta \mathbf{u}$ and $\delta \boldsymbol{\vartheta}$ are the test functions. To make the derivation of material and geometric stiffness matrices more convenient, the above expression is further rewritten in a more compact form [21]:

$$G(\mathbf{r}, \mathbf{\Lambda}, \delta \boldsymbol{\Phi}) = \int_0^L \left[\left(\boldsymbol{\Xi}^T \delta \boldsymbol{\Phi} \right)^T \boldsymbol{\Pi} (\mathbf{H}_b + \mathbf{H}_c) - \delta \boldsymbol{\Phi}^T \mathbf{p}_{\text{dis}} \right] dS \quad (13)$$

where the differential operator $\boldsymbol{\Xi}$ is defined as

$$\boldsymbol{\Xi}^T \equiv \begin{bmatrix} \frac{d}{ds} \mathbf{I} (\mathbf{r}')^\wedge \\ \mathbf{0} \quad \frac{d}{ds} \mathbf{I} \end{bmatrix}, \quad \frac{d}{ds} \mathbf{I} \equiv \text{diag} \left[\frac{d}{ds} \quad \frac{d}{ds} \quad \frac{d}{ds} \right] \quad (14)$$

and

$$\begin{aligned} \mathbf{H}_b &\equiv \begin{bmatrix} \mathbf{F}_b \\ \mathbf{M}_b \end{bmatrix}, \quad \mathbf{H}_c \equiv \begin{bmatrix} \mathbf{F}_c \\ \mathbf{M}_c \end{bmatrix}, \quad \mathbf{p}_{\text{dis}} \equiv \begin{bmatrix} \mathbf{f}_{\text{dis}} \\ \mathbf{m}_{\text{dis}} \end{bmatrix} \\ \delta \boldsymbol{\Phi} &\equiv \begin{bmatrix} \delta \mathbf{u} \\ \delta \boldsymbol{\vartheta} \end{bmatrix}, \quad \boldsymbol{\Pi} \equiv \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix} \end{aligned} \quad (15)$$

We follow the same approach of Simo and Vu-Quoc and use the following equation

$$\mathbf{\Lambda}(\boldsymbol{\vartheta}) \equiv \exp(\boldsymbol{\vartheta}^\wedge) \tilde{\mathbf{\Lambda}} \quad (16)$$

for the configuration update and derivation of tangent stiffness matrix, where $\tilde{\mathbf{\Lambda}}$ denotes the rotation tensor at a previous instant in time or loading history, for example, the configuration of the last iteration in an iterative method of solving the nonlinear equations. The variable $\boldsymbol{\vartheta}$ denotes the incremental rotation following $\tilde{\mathbf{\Lambda}}$.

To perform the linearization of the weak form at a configuration denoted by $\{\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}\}$, we define a displacement-rotation field $\boldsymbol{\Phi} = \begin{bmatrix} \mathbf{u} \\ \boldsymbol{\vartheta} \end{bmatrix}$ and a differential change $d\boldsymbol{\Phi} = \begin{bmatrix} d\mathbf{u} \\ d\boldsymbol{\vartheta} \end{bmatrix}$ at the configuration $\{\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}\}$. Based on the updating scheme of Eq. (16), the associated differential change for the rotation tensor is

$$d\mathbf{\Lambda} = \exp(d\boldsymbol{\vartheta}^\wedge) \tilde{\mathbf{\Lambda}} - \tilde{\mathbf{\Lambda}} = d\boldsymbol{\vartheta}^\wedge \tilde{\mathbf{\Lambda}} \quad (17)$$

The linearization of $\boldsymbol{\Gamma}$ may be expressed as

$$d\boldsymbol{\Gamma} = \tilde{\mathbf{\Lambda}}^T d\mathbf{u}' + d(\mathbf{\Lambda}^T) \tilde{\mathbf{r}}' = \tilde{\mathbf{\Lambda}}^T [d\mathbf{u}' + (\tilde{\mathbf{r}}')^\wedge d\boldsymbol{\vartheta}] \quad (18)$$

The linearization of $\boldsymbol{\kappa}$ is given by

$$d\boldsymbol{\kappa} = \left[d(\mathbf{\Lambda}^T) \tilde{\mathbf{\Lambda}}' \right]^\vee + \left[\tilde{\mathbf{\Lambda}}^T (d\mathbf{\Lambda})' \right]^\vee = \tilde{\mathbf{\Lambda}}^T d\boldsymbol{\vartheta}' \quad (19)$$

Using Eq. (18) and Eq. (19), we obtain

$$\left. \frac{\partial \mathbf{H}_b}{\partial \boldsymbol{\Phi}} \right|_{\boldsymbol{\Phi}=0} = \tilde{\mathbf{C}} \tilde{\boldsymbol{\Pi}}^T \tilde{\boldsymbol{\Xi}}^T \quad (20)$$

The linear part of the weak form (i.e., Eq. 13) at the configuration $\{\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}\}$ can be written as [21]

$$L[G] = G(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}, \delta \boldsymbol{\Phi}) + \Delta G_1 + \Delta G_2 + \Delta G_3 + \Delta G_4 \quad (21)$$

where ΔG_1 , ΔG_2 , ΔG_3 , and ΔG_4 are associated with the material, tendon loading, geometric, and external loading parts of the functional, respectively, and can be written as

$$\Delta G_1 = \int_0^L \left[\left(\tilde{\Xi}^T \delta \Phi \right)^T \tilde{\Pi} \left(\frac{\partial \mathbf{H}_b}{\partial \Phi} \Big|_{\Phi=0} \Delta \Phi \right) \right] dS = \int_0^L \left(\tilde{\Xi}^T \delta \Phi \right)^T \tilde{c} \tilde{\Xi}^T \Delta \Phi dS \quad (22)$$

$$\Delta G_2 = \int_0^L \left[\left(\tilde{\Xi}^T \delta \Phi \right)^T \tilde{\Pi} \left(\frac{\partial \mathbf{H}_c}{\partial \Phi} \Big|_{\Phi=0} \Delta \Phi \right) \right] dS \quad (23)$$

$$\Delta G_3 = \int_0^L \left[\left(\tilde{\mathbf{H}}_b + \tilde{\mathbf{H}}_c \right)^T \frac{\partial \left(\Pi^T \Xi^T \delta \Phi \right)}{\partial \Phi} \Big|_{\Phi=0} \Delta \Phi \right] dS = \int_0^L \left(\tilde{\Psi}^T \delta \Phi \right)^T \left(\tilde{\mathbf{B}} \tilde{\Psi}^T \Delta \Phi \right) dS \quad (24)$$

$$\Delta G_4 = - \int_0^L \left(\delta \Phi^T \frac{\partial \mathbf{p}_{\text{dis}}}{\partial \Phi} \Big|_{\Phi=0} \Delta \Phi \right) dS \quad (25)$$

In the above equations, $\Delta \Phi = \begin{bmatrix} \Delta \mathbf{u} \\ \Delta \vartheta \end{bmatrix}$ is the incremental displacement-rotation field; $\mathbf{c}$, Ψ , and $\mathbf{B}$ are defined as [21]

$$\mathbf{c} = \Pi \mathbf{C} \Pi^T \quad (26)$$

$$\Psi = \begin{bmatrix} \frac{d}{ds} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{d}{ds} \mathbf{I} & \mathbf{I} \end{bmatrix} \quad (27)$$

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & -(\mathbf{f}_b + \mathbf{f}_c)^\wedge \\ \mathbf{0} & \mathbf{0} & -(\mathbf{m}_b + \mathbf{m}_c)^\wedge \\ (\mathbf{f}_b + \mathbf{f}_c)^\wedge & \mathbf{0} & (\mathbf{f}_b + \mathbf{f}_c) (\mathbf{r}^T)' - (\mathbf{f}_b + \mathbf{f}_c)^T \mathbf{r}' \mathbf{I} \end{bmatrix} \quad (28)$$

The compact forms in Eqs. (22), (24), and (25) were first presented in the work by Simo and Vu-Quoc [18, 21].

2.4. Linearization of the Tendon Force and Couple Vectors

As shown in Eq. (23), for the tendon-driven continuum robots, the derivative $\frac{\partial \mathbf{H}_c}{\partial \Phi} \Big|_{\Phi=0}$ needs to be derived. For differential change $d\Phi$ to the configuration $\{\tilde{\mathbf{r}}, \tilde{\Lambda}\}$, we have

$$d\mathbf{b} = \tilde{\mathbf{Y}} d\chi' = \tilde{\mathbf{Y}} \left\{ (d\mathbf{u})' + \left[(d\vartheta^\wedge)' \tilde{\Lambda} + (d\vartheta^\wedge) \tilde{\Lambda}' \right] \tilde{\Lambda}^T \tilde{\zeta} \right\} \quad (29)$$

where $\mathbf{Y} = \frac{(\mathbf{I} - \mathbf{b}\mathbf{b}^T)}{|\chi'|}$. Using Eq. (9) and Eq. (10), the differential change to $\mathbf{F}_c$ can be derived as

$$d\mathbf{F}_c = T_c d(\Lambda^T \mathbf{b}) = T_c \tilde{\Lambda}^T \left[\tilde{\mathbf{Y}} \frac{d}{ds} \mathbf{I}, -\tilde{\mathbf{Y}} \left(\tilde{\zeta}^\wedge \frac{d}{ds} \mathbf{I} + (\tilde{\Omega} \tilde{\zeta})^\wedge \right) + \tilde{\mathbf{b}}^\wedge \right] \begin{bmatrix} d\mathbf{u} \\ d\vartheta \end{bmatrix} \quad (30)$$

The differential change to $\mathbf{M}_c$ can be expressed as

$$d\mathbf{M}_c = (\Lambda^T \zeta)^\wedge d\mathbf{F}_c = T_c (\tilde{\Lambda}^T \zeta)^\wedge \tilde{\Lambda}^T \left[\tilde{\mathbf{Y}} \frac{d}{ds} \mathbf{I}, -\tilde{\mathbf{Y}} \left(\tilde{\zeta}^\wedge \frac{d}{ds} \mathbf{I} + (\tilde{\Omega} \tilde{\zeta})^\wedge \right) + \tilde{\mathbf{b}}^\wedge \right] \begin{bmatrix} d\mathbf{u} \\ d\vartheta \end{bmatrix} \quad (31)$$

The matrix Υ is defined as

$$\Upsilon = \Pi \frac{d\mathbf{H}_c}{d\Phi} = T_c \begin{bmatrix} \mathbf{Y} \frac{d}{ds} \mathbf{I}, & \mathbf{b}^\wedge - \mathbf{Y} \left(\zeta^\wedge \frac{d}{ds} \mathbf{I} + (\Omega \zeta)^\wedge \right) \\ \zeta^\wedge \mathbf{Y} \frac{d}{ds} \mathbf{I}, & \zeta^\wedge \mathbf{b}^\wedge - \zeta^\wedge \mathbf{Y} \left(\zeta^\wedge \frac{d}{ds} \mathbf{I} + (\Omega \zeta)^\wedge \right) \end{bmatrix} \quad (32)$$

where $(\Lambda^T \zeta)^\wedge = \Lambda^T \zeta^\wedge \Lambda$.

2.5. Finite Element Approximation

For a typical beam element I_e^h , the incremental displacement-rotation field $\Delta\Phi_e^h \equiv \begin{bmatrix} \Delta\mathbf{u}_e^h \\ \Delta\boldsymbol{\vartheta}_e^h \end{bmatrix}$ is interpolated in terms of shape functions as

$$\begin{aligned}\Delta\mathbf{u}_e^h(S) &= \sum_{I=1}^{n_{ne}} N_I(S) \Delta\mathbf{u}_I^h \\ \Delta\boldsymbol{\vartheta}_e^h(S) &= \sum_{I=1}^{n_{ne}} N_I(S) \Delta\boldsymbol{\vartheta}_I^h\end{aligned}\quad (33)$$

where n_{ne} is the number of nodes of the beam element; $N_I(S)$ is the shape function associated with node I ; $\Delta\mathbf{u}_I^h$ and $\Delta\boldsymbol{\vartheta}_I^h$ are the nodal displacement and rotation vector increments at node I , respectively. The linear part of the weak form (i.e., Eq. 13) related to element I_e^h at the configuration $\{\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}\}$ may be expressed as

$$L[G_e^h] = \left[\mathbf{S}_e^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) + \mathbf{T}_e^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) \right] \Delta\Phi_e^h + \mathbf{P}_e^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) \quad (34)$$

where $\{\mathbf{S}_e^h + \mathbf{T}_e^h\}$ is the tangent stiffness matrix of the element I_e^h at the configuration $\{\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}\}$, $\mathbf{S}_e^h$ is contributed by $\{\Delta G_1 + \Delta G_2\}$, and $\mathbf{T}_e^h$ is contributed by ΔG_3 . Let $\mathbf{S}_{eIJ}^h$ and $\mathbf{T}_{eIJ}^h$ denote the submatrices coupling node I and J in $\mathbf{S}_e^h$ and $\mathbf{T}_e^h$, respectively, from Eqs. (22) and (23) we have the expression

$$\mathbf{S}_{eIJ}^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) = \int_{I_e^h} \tilde{\mathbf{\Xi}}_I^h \mathbf{c}_e^h \tilde{\mathbf{\Xi}}_J^{hT} + \tilde{\mathbf{\Xi}}_I^h \tilde{\mathbf{Y}}_J^h dS \quad (35)$$

where $\mathbf{c}_e^h$ is the matrix form of a spatial elasticity tensor $\mathbf{c}_e^h = \mathbf{\Pi}^h \mathbf{C} \mathbf{\Pi}^{hT}$, $\tilde{\mathbf{\Xi}}_I^h$ is given by

$$\tilde{\mathbf{\Xi}}_I^h = \begin{bmatrix} N_I' \mathbf{I} & \mathbf{0} \\ -N_I(\tilde{\mathbf{r}}')^\wedge & N_I' \mathbf{I} \end{bmatrix} \quad (36)$$

and $\tilde{\mathbf{Y}}_I^h$ is given by

$$\tilde{\mathbf{Y}}_I^h = T_c \begin{bmatrix} N_I' \tilde{\mathbf{Y}}, & N_I \tilde{\mathbf{b}}^\wedge - \tilde{\mathbf{Y}} \left(N_I' \tilde{\boldsymbol{\zeta}}^\wedge + N_I (\tilde{\boldsymbol{\eta}} \tilde{\boldsymbol{\zeta}})^\wedge \right) \\ N_I' \tilde{\boldsymbol{\zeta}}^\wedge \tilde{\mathbf{Y}}, & N_I \tilde{\boldsymbol{\zeta}}^\wedge \tilde{\mathbf{b}}^\wedge - \tilde{\boldsymbol{\zeta}}^\wedge \tilde{\mathbf{Y}} \left(N_I' \tilde{\boldsymbol{\zeta}}^\wedge + N_I (\tilde{\boldsymbol{\eta}} \tilde{\boldsymbol{\zeta}})^\wedge \right) \end{bmatrix} \quad (37)$$

The geometric part of the tangent stiffness matrix $\mathbf{T}_{eIJ}^h$ takes the form

$$\begin{aligned}\mathbf{T}_{eIJ}^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) &= \int_{I_e^h} \mathbf{\Psi}_I^h \mathbf{B}_e^h(\mathbf{\Psi}_J^h)^T dS \\ &= \int_{I_e^h} \begin{bmatrix} \mathbf{0}, & -N_I' N_J (\tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_c)^\wedge \\ N_I N_J' (\tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_c)^\wedge, & -N_I' N_J (\tilde{\mathbf{m}}_b + \tilde{\mathbf{m}}_c)^\wedge + N_I N_J \left[(\tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_c) (\tilde{\mathbf{r}}^T)' - (\tilde{\mathbf{f}}_b + \tilde{\mathbf{f}}_c)^T \tilde{\mathbf{r}}' I \right] \end{bmatrix} dS\end{aligned}\quad (38)$$

The matrix $\mathbf{P}_e^h$ is the load vector or unbalanced force vector in element I_e^h [21], which is from $G(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}, \delta\Phi)$. Let $\mathbf{P}_{eI}^h$ denote the unbalanced nodal force in element I_e^h related to node I , it may be expressed as

$$\mathbf{P}_{eI}^h(\tilde{\mathbf{r}}, \tilde{\mathbf{\Lambda}}) = \int_{I_e^h} \left(\tilde{\mathbf{\Xi}}_I^h (\tilde{\mathbf{H}}_b + \tilde{\mathbf{H}}_c) - \begin{bmatrix} N_I \mathbf{I} & \mathbf{0} \\ \mathbf{0} & N_I \mathbf{I} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{f}}_{\text{dis}} \\ \tilde{\mathbf{m}}_{\text{dis}} \end{bmatrix} \right) dS \quad (39)$$

In the iterative procedure, the updating scheme is given by the formulae

$$\begin{aligned}\tilde{\mathbf{r}}_n^{i+1} &= \tilde{\mathbf{r}}_n^i + \Delta \mathbf{u}_n^i \\ \tilde{\mathbf{\Lambda}}_n^{i+1} &= \exp \left[(\Delta \boldsymbol{\vartheta}_n^i)^\wedge \right] \tilde{\mathbf{\Lambda}}_n^i\end{aligned}\quad (40)$$

where $(\cdot)_n^i$ denotes the variable of the i th iterative step in the n th incremental step.

2.6. Formulation for the Planar Problems

Under the circumstances that all the forces and the tendon path are in the same plane, the spatial problem is degenerated into a planar problem. Because the parameterization of rotation in the planar case is straightforward, i.e., only one parameter is needed, we derive the formulation specific for the planar case, which will be used as an alternative method to further verify our formulation and computer implementation.

For the robotic arm that lies along the z -axis ($\vec{e}_3$) at the initial configuration depicted in Fig. 3, the displacements of the robotic arm in the $\vec{e}_1$ and $\vec{e}_3$ directions are denoted by $u(S)$ and $w(S)$, respectively; $\theta(S)$ denotes the deflection angle of the cross section; and S denotes the initial arc length of the arm. As shown in Fig. 3, for the planar case, the orthogonal tensor $\bar{\mathbf{\Lambda}}(S)$ and position vector $\vec{r}(S)$ become

$$\mathbf{\Lambda} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \text{ and } \mathbf{r} = \begin{bmatrix} u \\ 0 \\ S + w \end{bmatrix} \quad (41)$$

Utilizing Eq. (4), the material strain measures may be expressed as

$$\mathbf{\Gamma} = \begin{bmatrix} -(1 + \frac{\partial w}{\partial S}) \sin \theta + \frac{\partial u}{\partial S} \cos \theta \\ 0 \\ (1 + \frac{\partial w}{\partial S}) \cos \theta + \frac{\partial u}{\partial S} \sin \theta - 1 \end{bmatrix} \text{ and } \boldsymbol{\kappa} = \begin{bmatrix} 0 \\ \frac{\partial \theta}{\partial S} \\ 0 \end{bmatrix} \quad (42)$$

Also, by using Eq. (9) and Eq. (10), the stress and tendon force resultants may be expressed as

$$\begin{aligned}\mathbf{f}_b &= \begin{bmatrix} [-(1 + w') \sin \theta \cos \theta + u' \cos^2 \theta] GA + [(1 + w') \sin \theta \cos \theta + u' \sin^2 \theta - \sin \theta] EA \\ 0 \\ [(1 + w') \sin^2 \theta - u' \sin \theta \cos \theta] GA + [(1 + w') \cos^2 \theta + u' \sin \theta \cos \theta - \cos \theta] EA \end{bmatrix} \\ \mathbf{f}_c &= \begin{bmatrix} T_c \sin \theta \\ 0 \\ T_c \cos \theta \end{bmatrix}\end{aligned}\quad (43)$$

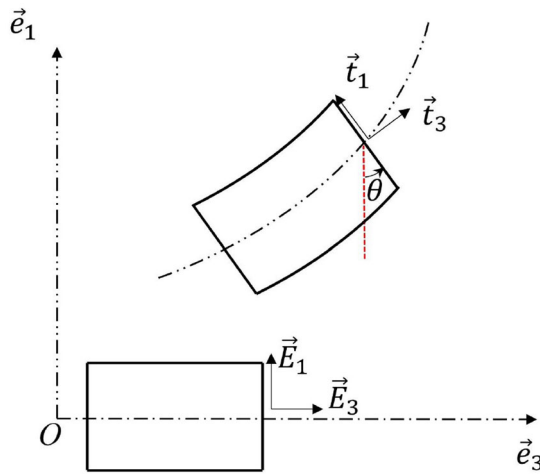


Fig. 3. The reference and current configurations of the planar problem

Accordingly, the stress couple may be expressed as

$$\mathbf{m}_b = \begin{bmatrix} 0 \\ EI \frac{\partial \theta}{\partial S} \\ 0 \end{bmatrix} \text{ and } \mathbf{m}_c = \begin{bmatrix} 0 \\ -T_c \zeta \\ 0 \end{bmatrix} \quad (44)$$

where ζ is the distance between the cable and the axis of the centroids, and $\mathbf{m}_c$ is constant with respect to S , i.e., $\frac{\partial \mathbf{m}_c}{\partial S} = 0$. For Eq. (11), by integrating the first formula and substituting the result into the second formula, we can obtain a simplified form of equilibrium equation as

$$\begin{aligned} \mathbf{f}_b(S) + \mathbf{f}_c(S) &= \mathbf{f}_b(L) + \mathbf{f}_c(L) + \int_S^L \mathbf{f}_{\text{dis}}(\tilde{S}) d\tilde{S} \\ \frac{d\mathbf{m}_b(S)}{dS} &= -\mathbf{m}_{\text{dis}}(S) - \left(\frac{d\mathbf{r}(S)}{dS} \right)^\wedge \left[\mathbf{f}_b(L) + \mathbf{f}_c(L) + \int_S^L \mathbf{f}_{\text{dis}}(\tilde{S}) d\tilde{S} \right] \end{aligned} \quad (45)$$

where L represents the initial length of the arm; and $\mathbf{f}_b$, $\mathbf{f}_c$, and $\mathbf{m}_b$ are given in Eq. (43) and Eq. (44). The simplified equilibrium equation of Eq. (45) for the planar case will be used to obtain analytical or semi-analytical solutions in the fourth numerical example in Sect. 3, where the boundary condition $\mathbf{f}_b(L) + \mathbf{f}_c(L)$ and the distributed loads $\mathbf{f}_{\text{dis}}(S)$ and $\mathbf{m}_{\text{dis}}(S)$ are given specifically. Note that if the end of the arm is free of forces, then $\mathbf{f}_b(L) + \mathbf{f}_c(L) = \mathbf{0}$.

3. Numerical Simulations

In this section, we present five numerical examples to illustrate the performance of the formulation described above. The elements used in the examples are two-node linear elements, and one-point Gaussian quadrature is employed to calculate the unbalanced nodal forces and tangent stiffness matrices. A full Newton-Raphson iterative solution procedure is employed in the numerical simulations, and the stopping criterion is

$$\sqrt{\frac{\sum_{I=1}^{N_n} \left[(\|\mathbf{u}_I\|_2)^2 + (\|R_g \boldsymbol{\vartheta}_I\|_2)^2 \right]}{\sum_{I=1}^{N_n} \left[\left(\left\| \sum_{\text{Previous steps}} \mathbf{u}_I \right\|_2 \right)^2 + \left(\left\| \sum_{\text{Previous steps}} R_g \boldsymbol{\vartheta}_I \right\|_2 \right)^2 \right]}} \leq \varepsilon \quad (46)$$

Here, ε is a prescribed tolerance, and in this study, we set $\varepsilon = 10^{-5}$; $\mathbf{u}_I$ and $\boldsymbol{\vartheta}_I$ are the iterative nodal translational and rotational displacements of node I , respectively; $\|\cdot\|_2$ represents the 2-norm; and R_g is a characteristic length to make the unit of rotational displacement term compatible with the translational displacement, and is defined by $R_g = \sqrt{J/A}$, where J and A are the polar moment of inertia and area of the cross section, respectively.

For the first two benchmark examples, the tendon tension is set to be zero, so the robot model reduces to the beam model developed by Simo and Vu-Quoc [21]. We compare our simulation results with the results published in the literature to verify our computer implementation. The last three examples are for the tendon-driven robotic arm. In these examples, we analyze the deformation of the tendon-driven robotic arm under several different external loads including tendon tension, weight, and spatial concentrated forces at the end of the arm. The model parameters in some of the examples are given as dimensionless quantities for ease of comparison with the literature.

3.1. Example 1: Pure Bending of a Cantilever Beam

This example, which has been investigated by Simo and Vu-Quoc [21], is used to validate the present implementation. A straight cantilever beam of length $L = 1$, which lies along the x -axis, is subjected to a concentrated bending moment M in the z -direction on its free end. Its material elasticity tensor components are $EA = GA_1 = GA_2 = 200$, and $EI_1 = EI_2 = \frac{1}{2}GJ = 1$, respectively. Since the exact solution to this problem can be easily obtained as $\vec{r} = \frac{EI}{M} \sin\left(\frac{MS}{EI}\right) \vec{e}_1 + \frac{EI}{M} \left(1 - \cos\left(\frac{MS}{EI}\right)\right) \vec{e}_2$ based on the planar case presented in Sect. 2.6, the deformed shape is a circular curve with radius

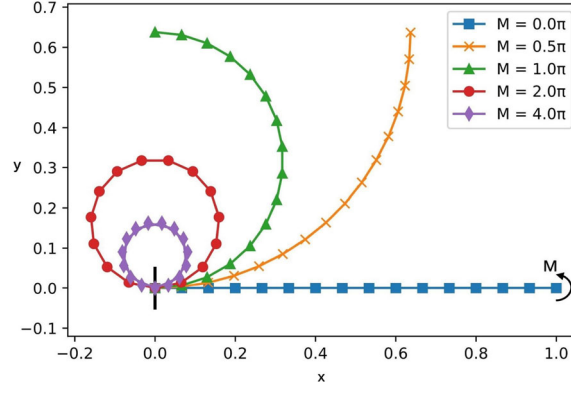


Fig. 4. Deformed shapes of pure bending of a cantilever beam subjected to different end moments

Table 1. Displacement components of selective points of the beam under end moment with different numbers of elements and corresponding exact solutions

Node's initial position $[x, y]$	[0.000, 0.000]	[0.200, 0.000]	[0.400, 0.000]	[0.600, 0.000]	[0.800, 0.000]	[1.000, 0.000]
Final position ($N_e = 5$)	[0.000, 0.000]	[0.062, 0.190]	[-0.100, 0.073]	[0.100, 0.073]	[-0.062, 0.190]	[0.000, 0.000]
Final position ($N_e = 10$)	[0.000, 0.000]	[0.050, 0.154]	[-0.081, 0.059]	[0.081, 0.059]	[-0.050, 0.154]	[0.000, 0.000]
Final position ($N_e = 15$)	[0.000, 0.000]	[0.048, 0.148]	[-0.078, 0.057]	[0.078, 0.057]	[-0.048, 0.148]	[0.000, 0.000]
Exact solution	[0.000, 0.000]	[0.047, 0.144]	[-0.076, 0.055]	[0.076, 0.055]	[-0.047, 0.144]	[0.000, 0.000]

Table 2. Displacement norms for different iteration and element numbers

Iteration number	Displacement norm ($N_e = 5$)	Displacement norm ($N_e = 10$)	Displacement norm ($N_e = 15$)
0	1.000	1.000	1.000
1	0.298	0.287	0.283
2	8.397e-15	1.521e-14	7.280e-15

$R = EI/M$. Accordingly, if the bending moment is chosen such that $M = 2n\pi$, the shape of the beam will be fully closed circles, and n is the number of times that the beam winds around itself. We focus on the situation when $n = 2$ and the number of elements are 5, 10, and 15. The computed results for displacement components of five equally selective points on the beam along with exact analytical solutions are shown in Table 1. The final shape of the beam discretized into 15 elements is also depicted in Fig. 4.

As indicated in Table 1, by increasing the number of elements, the numerical solution gradually approaches the exact solution. The displacement norms of each iteration can be found in Table 2. It can be seen that the convergency is achieved only by two iterations, as reported in [21, 25]. This is a particularity of this case, as commented in [25], which is a consequence of constant strains along the beam. Furthermore, we numerically observe that the condition $EA = GA_1 = GA_2$ must also be satisfied for the convergency to be achieved in two iterations.

3.2. Example 2: Cantilever 45-Degree Bend Subjected to Fixed-End Load

This example, which has already been analyzed by some researchers [21, 23, 33, 34], is employed to verify the accuracy of the presented formulation for 3D cases. A bend with the radius of 100 initially lies in the x - y plane and is fixed at one end. The concentrated tip load F is applied in the z -direction at the free end and deforms the shape of the bend, as shown in Fig. 5. In this case, the line of centroids

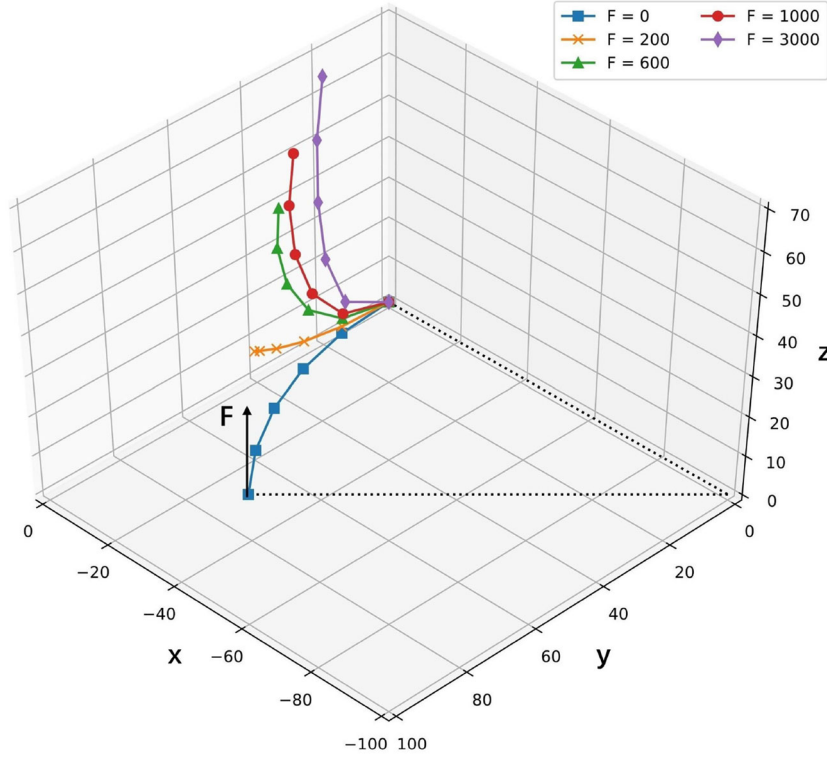


Fig. 5. Deformed shapes of the cantilever 45-degree bend subjected to different fixed-end loads

in the reference configuration is a curve, so the basis $\vec{E}_I$ becomes a function of S . The tensor $\bar{\Lambda}_0$ is defined as

$$\bar{\Lambda}_0 \equiv \vec{E}_I \otimes \vec{e}_I \quad (47)$$

Equation (5) will remain valid if $\bar{\Omega}$ is replaced by [18]

$$\bar{\Omega}_{\text{total}} \equiv \bar{\Omega} + \bar{\Lambda} \frac{\partial \bar{\Lambda}_0}{\partial S} \bar{\Lambda}_0^T \bar{\Lambda}^T \quad (48)$$

where the first term on the right-hand side denotes the “elastic curvature,” and the second term represents the “eigen curvature.”

The bend is discretized into eight elements, and the components of the material elasticity tensors are $EA = 2GA_1 = 2GA_2 = 10^7$, and $EI_1 = EI_2 = GJ = \frac{10^7}{12}$. The value of F is 600 and is applied gradually: 300, 450, and 600. The results of each load case for the components of the tip displacement (U, V, W) for the present study as well as [21, 24, 33] are given in Table 3, which show that the numerical results match well with those of the mentioned references. The iteration numbers needed for the convergency of results are also given in Table 4, indicating that the convergence history is similar to that of Simo and Vu-Quoc [21].

As already mentioned in the introduction section, Jelenic and Crisfield [24] showed that the interpolation method developed by Simo and Vu-Quoc [21] could be path-dependent. To check to what extent such dependency affects our numerical results, we consider three different loading paths: (i) the full load is applied in three steps, where half of the load in the first step and one quarter in each of the last two steps; (ii) the full load is equally divided into three loading steps; (iii) the load is equally divided into sixty loading steps. The components of the tip displacement of the bend obtained by the mentioned loading paths are displayed in Table 5. One can see from Table 5 that the path-dependency

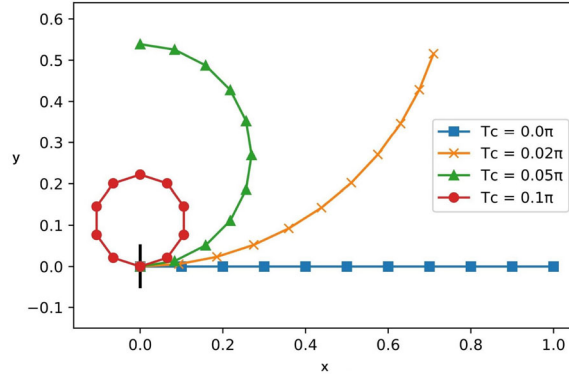


Fig. 6. Deformed shapes of tendon-driven continuum robot with different tendon tensions

is negligible and the relative variation of displacement results is less than 0.1%. Hence, the interpolation method adopted in the present work is adequate with an order of ten elements and produces satisfactory results for real engineering applications.

3.3. Example 3: Tendon-Contraction Causing the Continuum Robot to Bend

This example considers the effect of tendon on bending of the robotic arm similar to the work done by Camarillo et al. [3]. A tendon-driven robotic arm with a unit length lying in the x - y plane is fixed at its left end and free at the other end. The components of the material elasticity tensors for this arm are $EA = GA_1 = GA_2 = 1$, $EI_1 = EI_2 = \frac{1}{2}GJ = 0.005$. Initially, the cable is parallel with the central axis of the arm and has a distance $\xi = 0.1$ to the central axis. When the tendon tension T_c is applied, the straight arm deforms into an arc shape. The exact solution to this problem can be derived as a circular curve with radius $R = \frac{EI}{T_c \xi} \left(1 - \frac{T_c}{EA}\right)$. If the value of tendon tension is considered as $T_c = \frac{2n\pi EI}{\xi}$, the beam winds into circles by n times.

The robotic arms are discretized into 5 and 10 elements with the applied load of $T_c = 0.05\pi$. The deformation of the arm meshed with 10 elements is depicted for different values of applied tendon tensions in Fig. 6. It can be seen that by increasing the magnitude of T_c , the curvature of the arm increases. Table 6 summarizes the obtained results for six equally located points along the arm with two meshes ($N_e = 5, 10$) by the present formulation and exact solutions. It demonstrates that the coarse mesh already produces excellent results. The displacement norms of the mentioned results for each iteration are listed in Table 7. Unlike the first example where the convergence was achieved by only two iterations, here for the bending of tendon-driven continuum robots, the convergence requires more iterations even for the constant strain solution along the arm.

Table 3. Components of the tip displacement of the cantilever 45-degree bend

Load	Present			Simo et al. [21]			Bathe et al. [33]			Jelenic et al. [24]		
	U	V	W	U	V	W	U	V	W	U	V	W
300	6.99	-11.87	40.09	6.96	-11.87	40.08	6.79	-11.5	39.5	—	—	—
450	10.70	-18.40	48.40	10.67	-18.39	48.39	—	—	—	—	—	—
600	13.51	-23.48	53.37	13.50	-23.48	53.37	13.4	-23.5	53.4	13.48	-23.48	53.37

Table 4. Required iterations for the convergence

Load	Number of iterations (Present)	Number of iterations ([21])
300	13	13
450	7	8
600	5	6

Table 5. Components of the tip displacement of the 45-degree bend subjected to different loading paths

Load	U	W	V
i	13.51031	-23.47963	53.37487
ii	13.50433	-23.48112	53.37259
iii	13.48989	-23.48492	53.37834

Table 6. Displacement components of selective points of the arm under tendon tension with different numbers of elements and the corresponding exact solutions

Node's initial position $[x, y]$	[0.000, 0.000]	[0.200, 0.000]	[0.400, 0.000]	[0.600, 0.000]	[0.800, 0.000]	[1.000, 0.000]
Final position ($N_e = 5$)	[0.000, 0.000]	[0.160, 0.052]	[0.259, 0.188]	[0.259, 0.357]	[0.160, 0.493]	[0.000, 0.546]
Final position ($N_e = 10$)	[0.000, 0.000]	[0.158, 0.051]	[0.256, 0.186]	[0.256, 0.353]	[0.158, 0.487]	[0.000, 0.539]
Exact solution	[0.000, 0.000]	[0.158, 0.051]	[0.255, 0.185]	[0.255, 0.351]	[0.158, 0.485]	[0.000, 0.537]

Table 7. Displacement norms for different iteration and element numbers

Iteration number	Displacement norm ($N_e = 5$)	Displacement norm ($N_e = 10$)
0	1.000	1.000
1	0.766	0.777
2	0.238	0.235
3	0.456	0.454
4	0.187	0.178
5	0.126	0.133
6	0.017	0.019
7	0.006	0.006
8	3.59e-5	3.63e-5
9	3.51e-8	6.83e-8

3.4. Example 4: Tendon-Driven Continuum Robot Lifting a Weight

In this example, we apply the computational model to simulate the deformation of a tendon-driven robotic arm when lifting a weight at its end, which is a common task for robotic arms. An experiment is also conducted so that we can compare the simulation results with those of the experiment. As shown in the bottom row of Fig. 7, a soft continuum robotic arm is aligned along the vertical direction, with its top end fixed. A weight is attached to the center of the bottom surface. A force gauge (not shown in the figure) pulls the tendon and causes the robotic arm to bend. The centroid line of the robotic arm is marked black for extracting the deformed robot configuration. The parameters used in the simulations are set to be the same as those of the experiment. The length of the robotic arm is $L = 90$ mm, the distance between the tendon and the line of centroids is $\xi = 9$ mm, Young's modulus $E = 0.1$ N/mm², and Poisson's ratio $\nu = 0.5$ for the silicon rubber. For the rectangular shape of the cross section of the robotic arm, the components of the material elasticity tensors for this arm can be determined as $EA = 3GA_1 = 3GA_2 = 40.796$ N, $EI_1 = 1364.973$ N mm², $EI_2 = 2609.279$ N mm² and $GJ = 1324.751$ N mm². The self-weight per unit length of the robotic arm is $w_g = 0.004508$ N/mm, and the weight applied at the bottom end is $p = 2.0$ N. For both the experiment and the simulation, in a quasi-static process, a series of tendon forces are applied and the deformed configurations of the arm are simulated or measured. As shown in Fig. 7, when the tendon tension is less than about 4 N, the simulation results are in a good agreement with those of the experiment. As the tendon tension further increases, the differences between the simulation results and the experimental results also increase, for example, for $T_c = 5.14$ N, the end position difference is about 4.18 mm, which is about 25.0% of the total displacement of the end. There are two possible reasons for the discrepancy: (1) The kinematic assumption of the rigid cross section is not appropriate when the tendon tension T_c is large, because of the apparent radial expansion, as shown in Fig. 7d. (2) The linear force-strain relation used in the simulations may not be accurate enough.

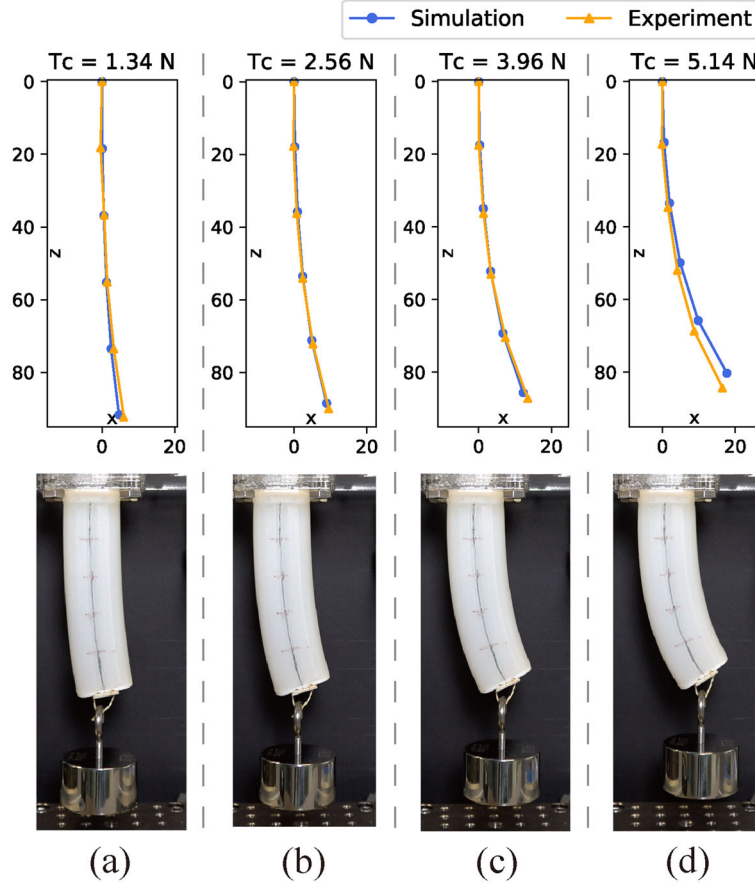


Fig. 7. The simulation and experimental results of the deformation of the robotic arm when lifting a weight

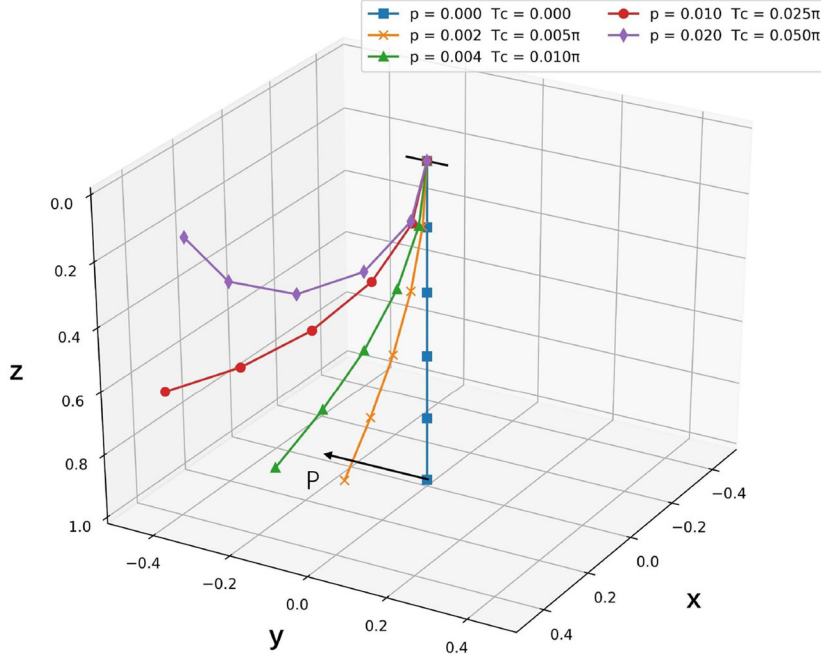
Next, we apply the formulation appeared in Sect. 2.6 for the planar case to solve this problem. In this example, $\mathbf{f}_{\text{dis}} = \begin{bmatrix} 0 \\ 0 \\ w_g \end{bmatrix}$, $\mathbf{m}_{\text{dis}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{f}_b(L) + \mathbf{f}_c(L) = \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{m}_b(L) + \mathbf{m}_c(L) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, and the arm is fixed at $S = 0$ and free at the end. The equilibrium equations may take the form

$$\begin{cases} \mathbf{f}_b(S) + \mathbf{f}_c(S) = \begin{bmatrix} 0 \\ 0 \\ w_g(L - S) + p \end{bmatrix} & S \in (0, L) \\ \frac{d\mathbf{m}_b(S)}{dS} = \begin{bmatrix} 0 \\ w_g(L - S)u' + pu' \\ 0 \end{bmatrix} & S \in (0, L) \\ u = w = \theta = 0 & S = 0 \\ \frac{d\theta(S)}{dS} = \frac{T_c \zeta}{EI} & S = L \end{cases} \quad (49)$$

Substituting $\mathbf{f}_b$, $\mathbf{f}_c$ and $\mathbf{m}_b$ from Eq. (43) and Eq. (44) into Eq. (45), the ordinary differential equations (i.e., Eq. (49)) of u , w , and θ can be obtained. Equation (49) has exact solutions if the robotic arm's weight is the unique load acting on the arm; in this case, the configuration of the deformed arm can be described by: $u = \theta = 0$ and $w = \frac{w_g LS}{EA} - \frac{w_g S^2}{2EA}$. But in the case of tendon tension and weights acting on the arm simultaneously, the exact solutions cannot be obtained, so we use the shooting method

Table 8. Deformed position of the equally distributed points along the arm when lifting a weight

Node's initial position $[x, z]$	[0.0, 0.0]	[0.0, 18.0]	[0.0, 36.0]	[0.0, 54.0]	[0.0, 72.0]	[0.0, 90.0]
Final position (GEB theory-based FEM)	[0.0, 0.0]	[0.5, 16.8]	[2.0, 34.4]	[4.9, 49.9]	[9.8, 65.8]	[17.7, 80.3]
Final position (experiment)	[0.0, 0.0]	[0.0, 17.3]	[1.5, 34.7]	[4.1, 52.0]	[8.7, 68.6]	[16.5, 84.3]
Final position (shooting method)	[0.0, 0.0]	[0.5, 16.8]	[2.0, 33.4]	[5.0, 49.9]	[9.9, 65.7]	[17.7, 80.3]


 Fig. 8. The deformed configurations of the tendon-driven continuum robot under spatial loads, p is the point load, T_c is the tendon tension

to solve Eq. (49) numerically. The results of the shooting method, the GEB theory-based FEM, and the experiment, for the scenario shown in Fig. 7d, are listed in Table 8. It is evident that the GEB theory-based FEM has similar accuracy as the shooting method. From both Fig. 7 and Table 8, one can see that the axial shortening in the simulation is larger than that of the experiment, which may be because that the axial expansion is not considered in the GEB theory-based FEM simulation.

3.5. Example 5: Tendon-Driven Continuum Robot under Spatial Free End Load

In this example, the external load p is applied along the negative y -direction ($\vec{e}_2$) and the robotic arm's own weight is ignored. In this case, since the load is not applied in the plane containing the line of centroids and tendon line, the problem becomes spatial. The tendon-driven robotic arm has the same properties as those used in Sect. 3.3. As shown in Fig. 8, the deformations of the arm in x - y - z coordinates under different values of p and T_c are obtained by applying p and T_c simultaneously. The obtained locations of the equally distributed points on the arm are also listed in Table 9 for each loading case. In this example, in addition to bending and axial compression, the robotic arm undergoes torsion because of the spatial loading.

4. Conclusion

In this work, we have formulated a geometrically nonlinear structural finite element model for tendon-driven continuum robots. The continuum robotic arm is modeled using the geometrically exact beam theory, and the embedded tendon inside the arm is considered as a part of the arm. Therefore, only tendon tension and its moment appear in the equilibrium equation, which results in simpler formulation than previous computational models that treated the tendon as an external object to the

Table 9. Deformed position of the equally distributed points along the arm subject to the spatial loading

Initial position	$p = 0.002$ $T_c = 0.005\pi$	$p = 0.004$ $T_c = 0.010\pi$	$p = 0.010$ $T_c = 0.025\pi$	$p = 0.020$ $T_c = 0.050\pi$
[0.000, 0.000, 0.000]	[0.000, 0.000, 0.000]	[0.000, 0.000, 0.000]	[0.000, 0.000, 0.000]	[0.000, 0.000, 0.000]
[0.000, 0.000, 0.200]	[0.006, -0.007, 0.197]	[0.012, -0.013, 0.193]	[0.027, -0.021, 0.181]	[0.051, -0.011, 0.161]
[0.000, 0.000, 0.400]	[0.025, -0.027, 0.392]	[0.047, -0.048, 0.380]	[0.102, -0.080, 0.340]	[0.177, -0.055, 0.266]
[0.000, 0.000, 0.600]	[0.055, -0.055, 0.584]	[0.104, -0.100, 0.558]	[0.214, -0.167, 0.460]	[0.327, -0.133, 0.283]
[0.000, 0.000, 0.800]	[0.097, -0.089, 0.773]	[0.183, -0.161, 0.725]	[0.353, -0.265, 0.533]	[0.451, -0.225, 0.210]
[0.000, 0.000, 1.000]	[0.151, -0.126, 0.959]	[0.281, -0.227, 0.878]	[0.511, -0.359, 0.555]	[0.514, -0.293, 0.067]

robot. The elastostatic finite element formulation has been derived and implemented in an in-house finite element code using Python language and object-oriented programming. We tested the code in several nonlinear elastostatic problems, which show satisfactory accuracy, efficiency, and generality of the formulation for tendon-driven continuum robots.

It should be noted that the present model assumes the rigid cross section of the robotic arm, which is well-suited for flexible continuum robots made of hard materials. However, it may be inaccurate for continuum robots made of soft elastomers that undergo significant area change in the cross section when the arm is being compressed by the tendon force. Another limitation of the current model when applied to soft robots is its linear force-strain relation, since the constitutive relations of soft materials in the range of large strain are usually nonlinear. Future studies could explore such issues by considering the area change of the cross section in the kinematics and the nonlinear constitutive relations of the materials for soft tendon-driven continuum robots.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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